INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 14, 529–537 (1992)

A SIMPLE AND EFFICIENT BEM ALGORITHM FOR PLANAR CAVITY FLOWS

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SUMMARY

This paper presents a boundary element formulation for solution of planar Riabouchinsky cavity flow problems. An iterative procedure for adjusting the free surface position is developed and shown to be stable and convergent. Numerical results are compared with finite difference and finite element solutions, showing the superior accuracy of the BEM models.

KEY WORDS Boundary element method Cavity flow Riabouchinsky flow Free surface flow

INTRODUCTION

The present paper considers the two-dimensional flow of an incompressible, inviscid fluid past a plate placed perpendicular to the flow direction in a channel of finite width and infinite length. Immediately behind the plate a cavity is formed containing air at a constant pressure; the primary objective of the study is to find the shape and size of this cavity.

It is known from experiments what sort of shape the cavity should have behind the plate. However, it is not clear at all, from experiments or theory, what the cavity form is in the region far from the plate. This phenomenon, known as the closure condition, is still open to discussion and several models have been presented in the literature.[1,2] In practice the downstream closure region is marked by considerable turbulence so that no potential flow solution can expect to be accurate in this region. However, most authors agree that within limits the choice of closure model has little effect on the flow at the upstream end of the cavity.[3]

Herein the model proposed by Riabouchinsky[4] is considered. The model assumes that an image plate can be placed in the flow at some point downstream from the original plate and that the flow geometry will be symmetric about the line midway between the plates (Figure 1). Since the flow is also symmetric about the channel axis, only one-quarter of the region in Figure 1 needs to be considered in the numerical analysis (Figure 2).

The determination of the free surface position CD is required as part of the solution of the problem. This generally involves an initial guess of this free streamline and the application of a systematic shifting algorithm to improve its location.

Previous attempts using the finite difference method (FDM)[5] and the finite element method (FEM)[6] presented many difficulties owing to the need to mesh the entire flow domain. The presence of a (mild) singularity at the separation point C requires the mesh to be very refined near this point and coarser elsewhere, i.e. a variable grid spacing.

0271–2091/92/050529–09$05.00

Received February 1991
Revised June 1991

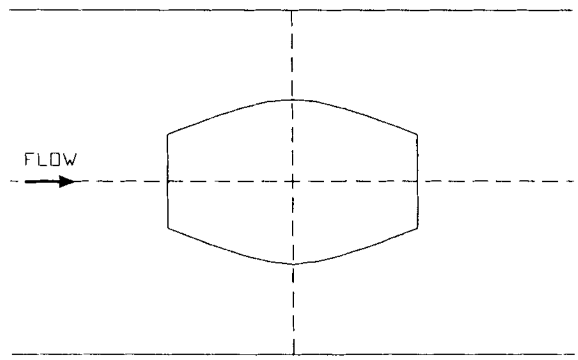

Figure 1. Riabouchinsky cavity model

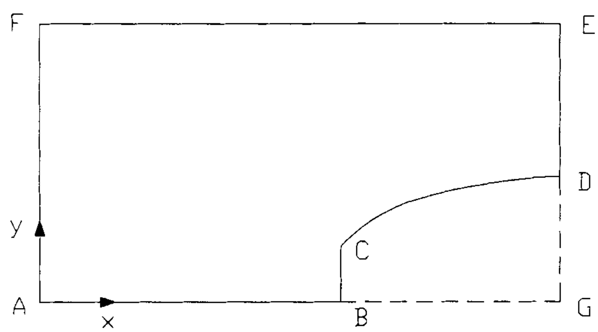

Figure 2. Definition of problem region

The most useful information from a solution of a cavity flow problem is the spatial location of the free surface and a knowledge of velocities and/or pressures along the boundaries. This realization leads one to feel that domain-type techniques are computationally inefficient and that the problem is ideally suited to a boundary element method (BEM) solution. This paper presents a BEM procedure in which linear elements are employed and the free surface position is determined iteratively. The iteration algorithm is a variant of the one employed by Cheng *et al.*[7] and Catabriga and Wrobel[8] for calculation of sluice gate flows.

A different BEM approach to the same problem was recently presented by Aitchison and Karageorghis[9] in which it is demonstrated that the determination of the free surface location is equivalent to the solution of a system of non-linear equations. Although the present algorithm is completely different in nature from that of Reference 9, the results obtained for some test problems are remarkably similar, as discussed at the end of this work.

FORMULATION OF THE PROBLEM

A sketch of the plane Riabouchinsky cavity model adopted in this work can be seen in Figure 2. The computational region is truncated at a certain distance upstream, where the flow is assumed to be parallel to the channel walls.

For the steady, irrotational, plane flow of an incompressible, inviscid fluid the following equations apply:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{(continuity)}, \tag{1}$$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0 \quad \text{(irrotationality)}, \tag{2}$$

$$p+\tfrac{1}{2}\rho q^2+\rho g y=\beta \quad \text{(Bernoulli)}, \tag{3}$$

in which u and v are the horizontal and vertical components of the velocity field whose magnitude is $q=(u^2+v^2)^{1/2}$, p is the pressure, ρ is the fluid density, g is the acceleration due to gravity and β is Bernoulli's constant.

Introducing a streamfunction ψ such that

$$u=-\frac{\partial\psi}{\partial y}, \qquad v=\frac{\partial\psi}{\partial x}, \tag{4}$$

and substituting into (1) and (2), one can note that the continuity equation is identically satisfied and the irrotationality condition gives rise to Laplace's equation

$$\nabla^2\psi=0. \tag{5}$$

The boundary conditions of the problem are as follows, referring to Figure 2. Line ABCD is a streamline and without loss of generality its value can be set to $\psi=0$. The channel wall, line EF, is also a streamline with value $\psi=Q$, where Q is the flow rate per unit width. Line DE is a symmetry axis, so $\partial\psi/\partial n=0$ can be applied. At the truncating boundary AF (of length H) the onset velocity is constant; thus it is possible to apply either $\psi=Qy/H$ or $\partial\psi/\partial n=0$.

The problem is solved by assuming an initial guess for the free surface position and using Bernoulli's equation to correct it during an iteration process. It is customary to assume that the hydrostatic pressure is approximately uniform over the region of interest and to neglect the effects of gravity; as a consequence of (3) the fluid velocity is uniform in magnitude on the free streamline. This is an accurate assumption when $gy \ll q^2$, which holds true in this case.[10]

Denoting quantities referenced to the cavity or free streamline by the subscript 'c', we may write from (3)

$$p_e+\tfrac{1}{2}\rho q^2=\tfrac{1}{2}\rho q_c^2=\beta, \tag{6}$$

in which $p_e=p-p_c$ and the constant value of the pressure p_c has been incorporated into β. This expression permits computing the pressure at any point once the velocity field has been determined.

An important parameter in cavity flows is the Prandtl cavitation number σ defined by

$$\sigma=\frac{q_c^2}{q_0^2}-1, \tag{7}$$

where q_0 is the onset velocity magnitude ($q_0=Q/H$).

The problem can be made non-dimensional by putting

$$x^*=x/H, \qquad y^*=y/H, \qquad \psi^*=\psi/Q. \tag{8}$$

The mathematical problem to be solved is then described by the equations (dropping the asterisks for simplicity)

$$\nabla^2\psi=0 \qquad \text{in ABCDEFA}, \tag{9}$$

$$\psi = 0 \qquad \text{on AB, BC, CD,} \tag{10}$$

$$\frac{\partial \psi}{\partial x} = 0 \qquad \text{on DE,} \tag{11}$$

$$\psi = 1 \qquad \text{on EF,} \tag{12}$$

$$\frac{\partial \psi}{\partial x} = 0 \qquad \text{on AF,} \tag{13}$$

$$\frac{\partial \psi}{\partial n} = -q_c \qquad \text{on CD,} \tag{14}$$

with the cavitation number given by

$$\sigma = q_c^2 - 1. \tag{15}$$

NUMERICAL SOLUTION

For the numerical solution of the problem an initial free surface location has to be assumed. The boundaries of the region can then be discretized and the problem solved using a standard BEM methodology.[11]

In the present case, however, advantage was taken of the problem symmetry by using a fundamental solution which implicitly satisfies the symmetry conditions along the lines AB and DE in the form

$$u^* = \frac{1}{2\pi} \ln\left(\frac{r_2 r_3}{r_1 r_4}\right),$$

where

$$r_1 = \{[x(\xi) - x(\chi)]^2 + [y(\xi) - y(\chi)]^2\}^{1/2},$$
$$r_2 = \{[x(\xi) - x(\chi)]^2 + [y(\xi) + y(\chi)]^2\}^{1/2},$$
$$r_3 = \{[x(\xi) + x(\chi) - 2l]^2 + [y(\xi) - y(\chi)]^2\}^{1/2},$$
$$r_4 = \{[x(\xi) + x(\chi) - 2l]^2 + [y(\xi) + y(\chi)]^2\}^{1/2},$$

in which ξ and χ are the source and field points respectively and $l = \overline{AG}$. Thus only the lines BC, CD, EF and FA need to be discretized, reducing the total number of degrees of freedom of the problem.

The boundary condition applied on the free streamline CD is $\psi = 0$. The BEM solution then produces the value of $\partial\psi/\partial n$ at each point along the free surface. These values are all equal for the correct free surface position (but of course not for the assumed one) and it is essential that a procedure be devised to properly move the free surface in the steps of the iteration scheme so that the constant-velocity condition is more closely satisfied on the moved streamline.

With the objective of obtaining a relation between increments of q_c and y on the free surface points, we define $\bar{q}$ as the average velocity in a vertical section of the flow. Thus the flow rate (per unit width) can be expressed as

$$Q = \bar{q}(H - y) \tag{16}$$

or, in non-dimensional form,

$$\bar{q}(1 - y) = 1. \tag{17}$$

The next step is to derive an approximate relation between $\bar{q}$ and q_c. It is noted that in the region behind the plate $\bar{q}(x)$ should increase with x and be lower than the free surface velocity q_c. The following relation was then adopted:

$$\bar{q} = q_c \frac{y}{1-y}, \tag{18}$$

which complies with the previous requirements (at least for the range of problems studied). Substituting into expression (17), we obtain

$$q_c y = 1 \quad \text{or} \quad q_c = \frac{1}{y}. \tag{19}$$

The iteration procedure will then be as follows:

(i) Assume an initial free surface location.
(ii) The BEM solution provides $\partial\psi/\partial n$ along the free surface.
(iii) Calculate β at each free surface node i, i.e.

$$\beta_i = \tfrac{1}{2}\rho q_{ci}^2. \tag{20}$$

(iv) Calculate $\Delta\beta_i = \beta_c - \beta_i$, in which β_c is the value of β at point C, the separation point. This value was selected because point C, the plate tip, is a fixed free surface point.
(v) Calculate Δy_i by substituting (19) into (20) to obtain

$$\beta_i = \frac{\rho}{2y_i^2}, \tag{21}$$

$$\frac{\Delta\beta}{\Delta y} \approx \frac{d\beta}{dy} = -\frac{\rho}{y^3}, \tag{22}$$

$$\Delta y_i = -\frac{\Delta\beta_i y_i^3}{\rho}. \tag{23}$$

(vi) Compute the relative norm of increments

$$\sum_{i=1}^{N} \frac{1}{N}\left|\frac{y_i^{k+1} - y_i^k}{y_i^k}\right|, \tag{24}$$

where N is the number of free surface nodes and $y_i^{k+1} = y_i^k + \omega\Delta y_i^{k+1}$, ω being a relaxation coefficient. If the norm is smaller than a small tolerance ε, the process has converged and is terminated; otherwise the free surface is moved to its new location and the process returns to step (ii).

RESULTS OF ANALYSES

Some results are now presented concerning the problem depicted in Figure 2. For simplicity we make H and Q equal to unity and call $\overline{BC} = d$, $\overline{BG} = L$, $\overline{DG} = b$ and $\overline{AB} = a$.

For specified values of a and d the problem contains three free parameters, i.e. L, b and q_c, and one more needs to be prescribed to fully define the problem. As pointed out in Reference 6, the natural choice from the physical point of view is to fix q_c, which is equivalent to specifying the pressure far upstream. However, small changes in the pressure lead to large changes in the cavity

Table I. Results for $d=0{\cdot}1$, $L=1{\cdot}0$

Method	b	σ
BEM	0·269	1·039
FEM	0·226	0·812
FDM	0·297	2·070

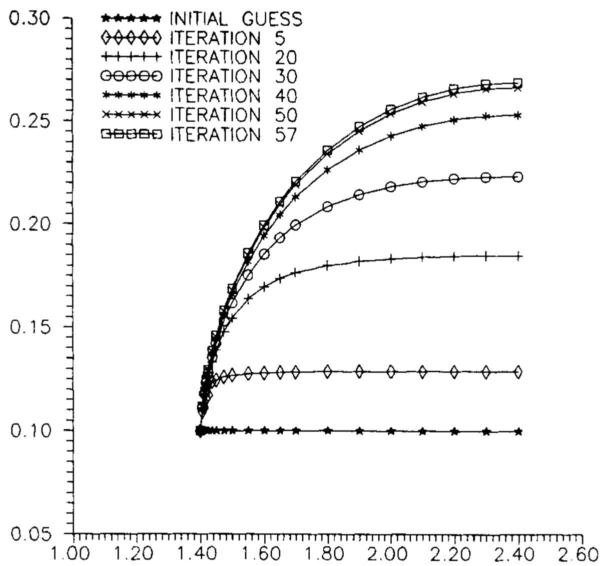

Figure 3. Convergence pattern for straight line as initial guess (distorted scale)

length and the problem becomes more difficult from the mathematical point of view. Thus it is preferable to fix the length L and compute b and q_c as part of the solution.

Results are presented in Table I for $d=0{\cdot}1$, $L=1{\cdot}0$ and a relaxation parameter $\omega=1{\cdot}0$. These results were obtained using two different initial guesses for the free surface: the straight line $y=d$ and a quarter of an ellipse with centre along the line $\overline{EG}$ at $y=d$ and semi-axes L and 0·10. Also presented are the FEM solution of Aitchison[6] and the FDM one of Mogel and Street.[5] It can be noticed that the results differ quite considerably. The FDM solution was obtained with a very refined mesh with 7450 grid points. The FEM procedure employed a refined moving mesh of linear triangular elements, computing the free surface position through a discrete minimization problem using a quasi-Newton method. This was solved repeatedly for different assumed values of q_c until the average pressure in all elements was positive. The BEM discretization employed 43 linear elements, with eight elements along the plate (line BC), 19 along the free surface (line CD), 12 along the channel wall (line EF) and four along the truncation boundary (line FA). No special analytical consideration was given to the singularity at the separation point C; the discretization, however, was graded in such a way that along the free surface the element size varied from 0·007 near point C to 0·1 near point D.

The BEM solution required a large number of iterations for a tolerance $\varepsilon=10^{-3}$. The convergence patterns for the two different initial free surface positions are depicted in Figures 3

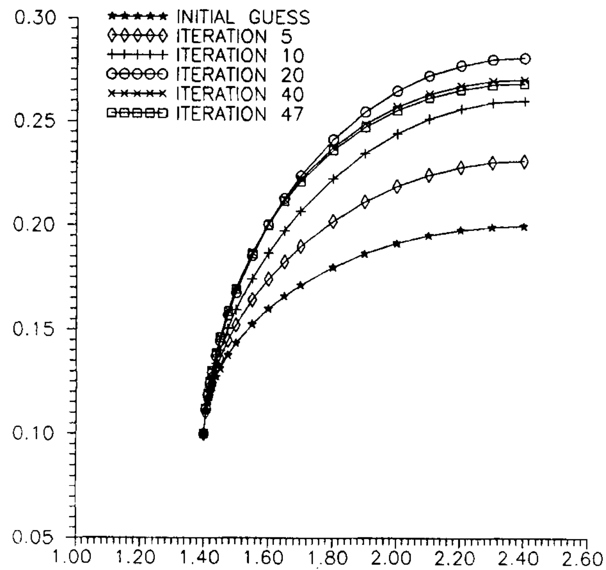

Figure 4. Convergence pattern for ellipse as initial guess (distorted scale)

Table II. Results for $d=0{\cdot}1$, $L=0{\cdot}5$

Method	a	b	σ
BEM	1	0·222	1·18
FEM	1	0·201	1·04
FDM	0·5	0·242	2·11
FDM	1	0·245	1·44
FDM	2	0·245	1·22

and 4. Convergence was achieved monotonically for the straight line; for the ellipse the free surface went above the correct one before it started bending towards the final position (notice results for iteration 20 in Figure 4). It can be seen that both cases converge to the same solution.

Table II depicts the results obtained by the three methods for the case $d=0{\cdot}1$, $L=0{\cdot}5$. The FDM appears to be very sensitive to the distance a of the truncating boundary, particularly in the calculation of σ. This was not the case for both the BEM and the FEM.

The case $d=0{\cdot}1$, $L=0{\cdot}5$ was also studied with a different BEM approach in Reference 9. Convergence of the method was verified by performing several analyses with increasing numbers of elements along the free streamline (from five to 25) and subsequently extrapolating the results to the limit. Also, the singularity at point C was taken into account by employing a curved element in the vicinity of this point. The converged results of Reference 9 are compared with the present ones in Table III.

Analytical results are only available for a channel of infinite width. The expressions are given by Birkhoff and Zarantonello[1] in terms of elliptic integrals and have been calculated numerically

Table III. BEM results for $d=0{\cdot}1$, $L=0{\cdot}5$

Method	b	σ
Present	0·222	1·18
Reference 9	0·220	1·16
Reference 9 with curved element	0·222	1·18

Table IV. Comparison with analytical solution for channel of infinite width

d	L	H/d	σ(BEM)	σ(FEM)	σ(FDM)
0·2	1·0	5	1·993	1·876	—
0·1	0·5	10	1·181	1·042	2·11
0·05	0·25	20	0·963	0·877	1·88
Analytic		∞		0·892	

Table V. Number of iterations (initial guess: ellipse)

d	L	No. of iterations	ω
0·2	1·0	21	0·25
0·1	0·5	28	1·00
0·05	0·25	33	4·00

by Aitchison.[6] Results in Table IV are for a channel with $L/d=5$ in which the effect of increasing the ratio of channel width to plate width is studied.

It can be seen that the FDM results are too high because of the small ratio a/d (compare the case $d=0{\cdot}1$, $L=0{\cdot}5$ with Table II). The FEM ones, on the other hand, are too low since for $H/d=20$ they are already lower than the analytical solution. The behaviour of the BEM results appears to be correct.

The iteration procedure in the BEM analysis can be speeded up by using a relaxation parameter ω different from 1·0. Table V shows the number of iterations required for the solutions presented in Table IV. In all cases the results obtained were exactly the same as for $\omega=1{\cdot}0$, showing the robustness of the scheme.

CONCLUSIONS

The BEM procedure for solving two-dimensional cavity flow problems described in the present paper has been shown to work well for a series of geometric configurations. The results obtained are superior to those of other numerical techniques, while the data preparation effort and computer time are much smaller.

The use of higher-order boundary elements (rather than linear) should lead to even more accurate results. In order to reduce the number of iterations, different forms for the approximation given by equation (18) are now being studied. Extension to axisymmetric cavity flow problems is also under way.

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